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The 30 Year Horizon

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New Foreword

On October 1, 2001 Axiom was withdrawn from the market and ended life as a commercial product. On September 3, 2002 Axiom was released under the Modified BSD license, including this document. On August 27, 2003 Axiom was released as free and open source software available for download from the Free Software Foundation's website, Savannah.

Work on Axiom has had the generous support of the Center for Algorithms and Interactive Scientific Computation (CAISS) at City College of New York. Special thanks go to Dr. Gilbert Baumslag for his support of the long term goal.

The online version of this documentation is roughly 1000 pages. In order to make printed versions we've broken it up into three volumes. The first volume is tutorial in nature. The second volume is for programmers. The third volume is reference material. We've also added a fourth volume for developers. All of these changes represent an experiment in print-on-demand delivery of documentation. Time will tell whether the experiment succeeded.

Axiom has been in existence for over thirty years. It is estimated to contain about three hundred man-years of research and has, as of September 3, 2003, 143 people listed in the credits. All of these people have contributed directly or indirectly to making Axiom available. Axiom is being passed to the next generation. I'm looking forward to future milestones.

With that in mind I've introduced the theme of the "30 year horizon". We must invent the tools that support the Computational Mathematician working 30 years from now. How will research be done when every bit of mathematical knowledge is online and instantly available? What happens when we scale Axiom by a factor of 100, giving us 1.1 million domains? How can we integrate theory with code? How will we integrate theorems and proofs of the mathematics with space-time complexity proofs and running code? What visualization tools are needed? How do we support the conceptual structures and semantics of mathematics in effective ways? How do we support results from the sciences? How do we teach the next generation to be effective Computational Mathematicians?

The "30 year horizon" is much nearer than it appears.

Tim Daly
CAISS, City College of New York
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Chapter 1

Implementation

1.1 Elementary Functions[4]

1.1.1 Rationale for Branch Cuts and Identities

Perhaps one of the most vexing problems to be addressed when attempting to determine a set of mathematical function definitions is the choice of the principal branches of the inverses of the exponential, trigonometric and hyperbolic functions, and, further, the mathematical form that these functions take on their domains (the complex plane slit by the corresponding branch cuts). The fundamental issue facing the mathematical library developer is the plethora of possibilities, and while some choices are demonstrably inferior, there is rarely a choice which is clearly best.

Following Kahan [1], we will refer to the mathematical formula we use to define the principal branch of each such function as its principal expression. For the inverse trigonometric and inverse hyperbolic functions, this principal expression is given in terms of the functions $\ln z$ and $\sqrt{z}$.

The choices set out in this Standard are derived from the following principles:

1. Branch cuts must lie completely within either the real or imaginary axis.
2. The principal expression must not have any singularities at finite points which the original function does not share.
3. Branch cuts end at branch points.
4. Where not otherwise determined, the value of a function on its branch cut or cuts is obtained by taking a limit along a path which approaches the branch cut in a counterclockwise manner around one of the branch points which terminate the cut (counterclockwise continuity, or CCC for short).
5. Each inverse trigonometric or hyperbolic function must be real-valued on the range

of the corresponding trigonometric or hyperbolic function when restricted to the real axis.

Further explanation of these principles can be found in [1].

While standard identities such as $\ln \frac{1}{x} = -\ln x$ hold for $x > 0$, they generally fail to hold for complex arguments of principal branches, even complex arguments which do not lie on a branch cut. Consequently, a definition of, say,

$$\arctan z = \frac{i}{2}(\ln(1 - iz) - \ln(1 + iz))$$

is not the same as the apparently equivalent

$$i \ln \left(\sqrt{\frac{1 - iz}{1 + iz}} \right)$$

. It can be challenging to decide if two candidate expressions for representing an inverse trigonometric or hyperbolic function which agree in the mathematical domain are the same in the restricted computational realm of principal expressions.

If the underlying computational mathematical system supports a signed zero, as prescribed by the IEEE/754 Standard [2], then a larger set of identities will hold. For example,

$$\ln \frac{1}{z} = -\ln z$$

holds for all complex z in such a system, as do conjugate symmetry relations for functions such as $\arcsin z$. However, identities such as $\ln zw = \ln z + \ln w$ still fail to hold for some complex z and w.

A useful function for representing identities involving complex functions which are related to the logarithm function is the complex signum function, defined as:

$$\operatorname{csgn}(z) = \begin{cases} 1, & \text{if } \Re z > 0 \text{ or } \Re z = 0 \text{ and } \Im z > 0 \\ -1, & \text{if } \Re z < 0 \text{ or } \Re z = 0 \text{ and } \Im z < 0 \end{cases}$$

The value of $\operatorname{csgn}(0)$ is unspecified. Note, for example, that $\sqrt{z^2} = z \operatorname{csgn}(z)$.

Using the principal expressions for each of the 12 inverse trigonometric and hyperbolic functions as given in this Standard, we have the following relations and identities:

1.1.2 Inverse trigonometric functions

$\arcsin(z)$	$= -\arcsin(-z)$ $= \frac{\pi}{2} - \arccos(z)$ $= -i\operatorname{arcsinh}(iz)$
$\arccos(z)$	$= \pi - \arccos(-z)$ $= \frac{\pi}{2} - \arcsin(z)$ $= i\operatorname{csgn}(i(z-1))\operatorname{arccosh}(z)$
$\arctan(z)$	$= -\arctan(-z)$ $= \frac{\pi}{2} - \operatorname{arccot}(z)$ $= -i\operatorname{arctanh}(iz)$ $= -i\ln\left(\frac{1+iz}{\sqrt{z^2+1}}\right)$
$\operatorname{arccot}(z)$	$= \pi - \operatorname{arccot}(-z)$ $= \frac{\pi}{2} - \arctan(z)$ $= i\operatorname{arccoth}(iz) + \frac{\pi}{2}(1 - \operatorname{csgn}(z+i))$ $= -i\ln\left(\frac{z+i}{\sqrt{z^2+1}}\right)$
$\operatorname{arccsc}(z)$	$= -\operatorname{arccsc}(-z)$ $= \arcsin(\frac{1}{z})$ $= \frac{\pi}{2} - \operatorname{arcsec}(z)$ $= i\ \operatorname{arccsch}(iz)$
$\operatorname{arcsec}(z)$	$= \pi - \operatorname{arcsec}(-z)$ $= \arccos(\frac{1}{z})$ $= \frac{\pi}{2} - \operatorname{arccsc}(z)$ $= i\operatorname{csgn}(i(\frac{1}{z} - 1))\operatorname{arcsech}(z)$

1.1.3 Inverse hyperbolic functions

$\mathrm{arcsin}h(z)$	$= -\mathrm{arcsinh}(-z)$ $= \frac{\pi}{2}i - \mathrm{csgn}(i-z)\mathrm{arccosh}(-iz)$ $= -i\mathrm{arcsin}(iz)$
$\mathrm{arccosh}(z)$	$= i\mathrm{csgn}(i(1-z))\mathrm{arccos}(z)$ $= \mathrm{csgn}(i(1-z))(\frac{\pi}{2}i - \mathrm{arcsinh}(iz))$
$\mathrm{arctanh}(z)$	$= -\mathrm{arctanh}(-z)$ $= \mathrm{arccoth}(z) - \frac{\pi}{2}i\mathrm{csgn}(i(z-1))$ $= -i\mathrm{arctan}(iz)$ $= -\ln\left(\frac{1-z}{\sqrt{1-z^2}}\right)$
$\mathrm{arccoth}(z)$	$= \mathrm{arctanh}(z) + \frac{\pi}{2}i\mathrm{csgn}(i(z-1))$ $= i\mathrm{arccot}(iz) + \frac{\pi}{2}i(\mathrm{csgn}(i(z-1))-1)$ $= i\mathrm{arctan}(-iz) + \frac{\pi}{2}i\mathrm{csgn}(i(z-1))$
$\mathrm{arccsch}(z)$	$= -\mathrm{arccscn}(-z)$ $= \mathrm{arcsinn}(\frac{1}{z})$ $= \mathrm{csgn}(i+\frac{1}{z})\mathrm{arcsech}(-iz) - \frac{\pi}{2}i$ $= i\mathrm{arccsc}(iz)$
$\mathrm{arcsech}(z)$	$= \mathrm{arccosh}(\frac{1}{z})$ $= i\mathrm{csgn}(i(1-\frac{1}{z}))\mathrm{arcsec}(z)$ $= \mathrm{csgn}(i(1-\frac{1}{z}))(\frac{\pi}{2}i + \mathrm{arccsch}(iz))$

Bibliography

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[2] IEEE standard 754-1985 for binary floating-point arithmetic, reprinted in ACM SIGPLAN Notices 22 #2 (1987), 9-25.

[3] IEEE standard 754-2008

[4] Numerical Mathematics Consortium Technical Specification 1.0 (Draft) `http://www.nmconstorium.org`