Knot or Unknot ?

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Contents

A Natural Question

Let A_n be the space of n *or fewer* disjoint arcs in the circle. In the special case n=0, we define A_0 to be the 2 improper arcs: the empty arc and the full circle. Thus A_0 has only 2 points, just like $\mathbb{S}^0 = \{-1, 1\}$. We have the inclusions:

$$A_0 \quad \subseteq \quad A_1 \quad \subseteq \quad A_2 \quad \subseteq \quad \cdots \quad \subseteq \quad A_n \quad \subseteq \quad A_{n+1} \quad \subseteq \quad \cdots$$

We make A_n into a metric space as follows. Let V and W be collections of disjoint arcs in A_n, so $V, W \subseteq \mathbb{S}^1$. Let $\mathbf{1}_V$ and $\mathbf{1}_W$ be the indicator functions for V and W. Define the metric $d(V,W) := \int_{\mathbb{S}^1} |\mathbf{1}_V - \mathbf{1}_W| d\theta$. So A_n can be regarded as a subset of $L^1(\mathbb{S}^1)$ and it inherits a metric from that. The metric can be viewed in another way; the symmetric difference of V and W is also a collection of arcs, and $d(V,W)$ is the total length of the arcs in this symmetric difference. The corresponding functions in the package are `arcsdistance()` and `arcssymmdiff()`.

In the **User Guide** vignette it is shown that there are homeomorphisms

$$A_n \quad \longleftrightarrow \quad \partial Z_n \quad \longleftrightarrow \quad \mathbb{S}^{2n} \tag{1}$$

where Z_n is the polar zonoid in $\mathbb{R}^{2n+1}$. These inclusions and homeomorphisms induce embeddings $\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$ in this large commutative diagram.

$$\begin{array}{ccccccccccccc}
A_0 & \subseteq & A_1 & \subseteq & A_2 & \subseteq & \cdots & \subseteq & A_n & \subseteq & A_{n+1} & \subseteq & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \\
\partial Z_0 & \hookrightarrow & \partial Z_1 & \hookrightarrow & \partial Z_2 & \hookrightarrow & \cdots & \hookrightarrow & \partial Z_n & \hookrightarrow & \partial Z_{n+1} & \hookrightarrow & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \\
\mathbb{S}^0 & \hookrightarrow & \mathbb{S}^2 & \hookrightarrow & \mathbb{S}^4 & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{S}^{2n} & \hookrightarrow & \mathbb{S}^{2n+2} & \hookrightarrow & \cdots
\end{array}$$

The embeddings in the middle row are induced from the inclusions on the top row. The middle row is almost surely a Whitney stratification, see [1], but I have not checked this. The embeddings in the bottom row are induce from those in the middle row. We will soon see that the embedding

$$\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2} \tag{2}$$

is *not* the standard one, which is formed by appending two zeros. The codimension is 2, which is the only codimension where knotting of spheres occurs (at least in the PL category, see [2]). So it is natural to ask:

Q: Is the embedding $\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$ unknotted ?

i.e. is it isotopic to the standard embedding $\mathbb{S}^{2n} \subseteq \mathbb{S}^{2n+2}$?

Unknot

In this section we show that the embedding (2) is *not* knotted. For the purposes of this vignette, we use the standard order for the basis of the trigonometric polynomials:

$$1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), ..., \cos(n\theta), \sin(n\theta) \qquad \theta \in [0, 2\pi] \tag{3}$$

and put 1 at the beginning instead of the end, as in the **polarzonoid** package.

This embedding is the composition:

$$\mathbb{S}^{2n} \;\; \to \;\; \partial Z_n \;\; \hookrightarrow \;\; \partial Z_{n+1} \;\; \to \;\; \mathbb{S}^{2n+2} \tag{4}$$

Focus first on the middle embedding, which is the composition:

$$\partial Z_n \;\; \to \;\; A_n \;\; \subseteq \;\; A_{n+1} \;\; \to \;\; \partial Z_{n+1} \tag{5}$$

If a point $p \in \partial Z_n$ maps to a set of disjoint arcs $a \in A_n$, then this composition (5) is:

$$p \;\; \mapsto \;\; \left(\; p, \int_a \cos((n+1)\theta) d\theta, \int_a \sin((n+1)\theta) d\theta \; \right) \tag{6}$$

And since a is a function of p, () can be written:

$$p \;\; \mapsto \;\; (\; p, \; v(p) \;) \qquad \text{for a function } v : \partial Z_n \to \mathbb{R}^2 \tag{7}$$

Returning now to (4), it is convenient to translate ∂Z_n and ∂Z_{n+1} so their centers are at 0. The center of ∂Z_n is $(\pi, 0, ..., 0)$ so only the first coordinate is changed, and (7) is still valid. Denoting the centered sets by adding a prime $'$, the composition is now

$$\mathbb{S}^{2n} \;\; \to \;\; \partial Z'_n \;\; \hookrightarrow \;\; \partial Z'_{n+1} \;\; \to \;\; \mathbb{S}^{2n+2} \tag{8}$$

After these translations, the first and last maps are simple multiplication and division by positive real functions, and (8) is:

$$u \quad \mapsto \quad (\ \alpha(u)u, v(\alpha(u)u)\ )\ /\ \left(\alpha^2(u) + |v(\alpha(u)u)|^2\ \right)^{1/2} \qquad \text{where} \ \ |u| = 1 \ \ \text{and} \ \ \alpha(u) > 0 \tag{9}$$

This can be simplified to

$$u \quad \mapsto \quad (\ \beta(u)u,\ w(u)\ ) \qquad \text{where} \ \ \beta(u) > 0 \ \ \text{and} \ \ w(u) \in \mathbb{R}^2 \ \ \text{and} \ \ |w(u)| < 1 \tag{10}$$

We are done if we can show:

Theorem: Any embedding $f : \mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$ that has the form

$$f(u) \ = \ (\ \beta(u)u,\ w(u)\ ) \qquad \text{where} \ \ \beta(u) > 0 \ \ \text{and} \ \ w(u) \in \mathbb{R}^2 \ \ \text{and} \ \ |w(u)| < 1 \tag{11}$$

is isotopic to the standard embedding.

Proof: First note that $\beta(u)$ and $w(u)$ are not independent of each other. In fact we have:

$$1 \ = \ |f(u)|^2 \ = \ \beta(u)^2 \cdot 1^2 \ + \ |w(u)|^2 \tag{12}$$

and so $\beta(u) = \left(1 - |w(u)|^2\right)^{1/2}$ and f depends only on $w : \mathbb{S}^{2n+2} \to \mathbb{R}^2$. Now $\mathbb{R}^2$ is contractible so w is homotopic to the 0 map. In fact, if we let the homotopy parameter be $t \in [0,1]$, we can define $w_t(u) := (1-t)w(u)$. Note that w_1 is the 0 map, and we always have $|w_t(u)| < 1$. So each intermediate w_t defines an intermediate f_t, which is an embedding. And the final f_1 is the standard embedding: $f_1(u) = (u, 0, 0)$. □

Some Sample Calculations

```
library( polarzonoid )
```

In this section, we verify that the induced embedding $\mathbb{S}^6 \hookrightarrow \mathbb{S}^8$ has the correct form (10), for a few test cases. Since the software functions put the constant term last (instead of first) the indexes for $\mathbb{S}^6$ are 1,2,3,4,5,6,9 (omitting 7 and 8).

```
idx = c(1:6,9)

#  make a random unit vector in S^6
set.seed(0)
u = rnorm(7) ;   u = u / sqrt( sum(u^2) )
# embed into S^8
up = spherefromarcs( arcsfromsphere(u), n=4 )
beta = up[idx] / u ; beta
```

```
## [1] 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536
```

```
range( diff(beta) )
```

```
## [1] -5.295764e-14  8.859580e-14
```

So all the relevant coordinates are scaled by the same number, up to numerical truncation.

Now repeat this test many times.

```r
count = 50
umat = array( rnorm(count*7), dim=c(count,7) )
umat = umat / sqrt( rowSums(umat^2) )
upmat = t( apply( umat, 1, function(u) { spherefromarcs( arcsfromsphere(u), n=4 ) } ) )
betamat = upmat[ ,idx] / umat
delta = apply( betamat, 1, diff )
range( delta )
```

```
## [1] -7.925771e-11  3.887302e-11
```

Once again, the relevant coordinates are scaled by the same number, up to numerical truncation.

References

[1] WIKIPEDIA CONTRIBUTORS. *Whitney conditions — Wikipedia, the free encyclopedia* [online]. 2022. Available at: https://en.wikipedia.org/w/index.php?title=Whitney_conditions&oldid=1119503349. [Online; accessed 3-June-2025]

[2] ZEEMAN, E. C. Unknotting combinatorial balls. *Annals of Mathematics*. 1963, **78**(3), 501–526.

Session Information

This document was prepared Tue Jun 10, 2025 with the following configuration:

```
R version 4.5.0 (2025-04-11 ucrt)
Platform: x86_64-w64-mingw32/x64
Running under: Windows 11 x64 (build 26100)

Matrix products: default
  LAPACK version 3.12.1

locale:
[1] LC_COLLATE=C
[2] LC_CTYPE=English_United States.utf8
[3] LC_MONETARY=English_United States.utf8
[4] LC_NUMERIC=C
[5] LC_TIME=English_United States.utf8

time zone: America/Los_Angeles
tzcode source: internal

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods   base

other attached packages:
[1] zonohedra_0.4-0   rgl_1.3.18        gifski_1.32.0-2   flextable_0.9.7
[5] polarzonoid_0.1-2
```

```
loaded via a namespace (and not attached):
 [1] katex_1.5.0            jsonlite_2.0.0         qpdf_1.3.5
 [4] compiler_4.5.0         pdftools_3.5.0         equatags_0.2.1
 [7] tinytex_0.57           Rcpp_1.0.14            zip_2.3.3
[10] xml2_1.3.8             magick_2.8.6           jquerylib_0.1.4
[13] fontquiver_0.2.1       systemfonts_1.2.3      textshaping_1.0.1
[16] uuid_1.2-1             yaml_2.3.10            fastmap_1.2.0
[19] R6_2.6.1               microbenchmark_1.5.0   gdtools_0.4.2
[22] curl_6.2.2             knitr_1.50             htmlwidgets_1.6.4
[25] logger_0.4.0           openssl_2.3.2          bslib_0.9.0
[28] rlang_1.1.6            V8_6.0.3               cachem_1.1.0
[31] xfun_0.52              sass_0.4.10            cli_3.6.5
[34] magrittr_2.0.3         digest_0.6.37          grid_4.5.0
[37] base64enc_0.1-3        askpass_1.2.1          lifecycle_1.0.4
[40] evaluate_1.0.3         glue_1.8.0             data.table_1.17.2
[43] fontLiberation_0.1.0   officer_0.6.8          ragg_1.4.0
[46] xslt_1.5.1             fontBitstreamVera_0.1.1 rmarkdown_2.29
[49] tools_4.5.0            htmltools_0.5.8.1
```